[Time: 3:00 Hrs.] **[Marks: 80]**

Please check whether you have got the right question paper.

N.B:
1. **All questions are compulsory.**
2. **Figures to the right indicate full marks.**
3. **Scientific calculator can be used.**

Q.1 a) Let μ be finitely additive set function defined on algebra $\mathcal{A}$. Show that μ is countably additive if and only if it has following property: If $A_n \in \mathcal{A}$ and $A_n \subset A_{n+1}$ for each positive integer n and if $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ then $\mu(\bigcup_{n\in\mathbb{N}} A_n) = \lim\limits_{n\to\infty} \mu(A_n)$. **10**

b) Attempt **any Two** of the following: **10**

i) Show that exterior (or outer) measure of an open rectangle in $\mathbb{R}^n$ is it's volume. **5**

ii) Construct a non-measurable subset of $\mathbb{R}$. **5**

iii) Show that closed subsets of $\mathbb{R}^d$ is measurable. **5**

Q.2 a) Suppose f is a nonnegative Lebesgue measurable function. Show that there exists an increasing sequence $\{\psi_n\}_{n=1}^{\infty}$ of non-negative simple Lebesgue measurable functions and such that $\lim\limits_{n\to\infty} \psi_n(x) = f(x)\ for\ all\ x.$ **10**

b) Attempt **any Two** of the following: **10**

i) Prove that the outer measure of countable set is zero. **5**

ii) State and prove Borel Cantelli lemma. **5**

iii) Prove or disprove: Every Lebesgue measurable function is continuous. **5**

Q.3 a) State and prove Fatou's lemma. Give an example to shows that strict inequality may hold in the conclusion. **10**

b) Attempt **any Two** of the following: **10**

i) State and prove Dominated Convergence Theorem. **5**

ii) Prove that a function $f: \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ is a Lebesgue measurable function if for every $a \in \mathbb{R}$, $f^{-1}((a, \infty])$ is Lebesgue measurable subset of $\mathbb{R}^d$. **5**

iii) Prove or disprove: Does there exist $f: [0,1] \to \mathbb{R}$, continuous function such that $f(x) = \chi[0, \frac{1}{2}](x)$ for almost every x, where χ is characteristic function. **5**

Q.4 a) Define signed measure on a measurable space X. Prove Hahn Decomposition Theorem of $\mathbb{R}^n$. **10**

b) Attempt **any Two** of the following: **10**

i) Show that the continuous function of compact support is dense in $\boldsymbol{L^1}(\mathbb{R}^{\boldsymbol{n}})$. **5**

ii) Define $f(x) = \frac{1}{x^{\frac{2}{3}}} \qquad 0 < x < 1$ **5**

$\qquad = 0 \qquad x = 0$

Show that f is Lebesgue integrable on $[0, 1]$ and $\int_0^1 \frac{1}{x^{\frac{2}{3}}}\, dx = 3$. Also find $f(x, 2)$.

iii) Show that a real valued function that is continuous on its measurable domain is measurable. **5**

****